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Application of the Chord Method to Obtain Analytical Expressions for Dancoff Factors in Stochastic Media

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Abstract—*In this paper the chord method is applied to the computation of Dancoff factors for doubly heterogeneous stochastic media, characteristic of prismatic and pebble bed designs of the Very High Temperature Gas-Cooled Reactor (VHTR), where TRISO fuel particles are randomly distributed in fuel compacts or fuel pebbles that are arranged in a full core configuration. Previous work has shown that a chord length probability distribution function (PDF) can be determined analytically or empirically and used to model VHTR lattices with excellent results. The key observation is that once the chord length PDF is known, Dancoff factors for doubly heterogeneous stochastic media can be expressed as closed-form expressions that can be evaluated analytically for infinite and finite media and semianalytically for a collection of finite media.*

Based on the assumption that the chord length PDF in the moderator region between two fuel kernels in a VHTR compact or pebble is exponential, which was shown to be an excellent approximation in previous work, closed-form expressions for Dancoff factors are derived for a range of configurations from infinite stochastic media to finite stochastic media, including multiple finite stochastic media in a background medium (e.g., a pebble bed core). Numerical comparisons with Monte Carlo benchmark results demonstrate that the closed-form expressions for the Dancoff factors for VHTR configurations are accurate over a range of packing fractions characteristic of prismatic and pebble bed VHTRs.

I. INTRODUCTION

The first study on the reduction in the resonance absorption for a single fuel lump due to neighboring fuel lumps was performed by Dancoff and Ginsburg[1] in 1944. They derived a formula in terms of multiple integrals over solid angles and surfaces. The reduction in the resonance absorption was later called the Dancoff factor. Preceding the work by Dancoff and Ginsburg, Dirac[2] calculated the neutron multiplication factor for a finite volume in 1943, where he introduced the chord method to replace the complicated integrals over angles and surfaces with one-dimensional (1-D) integrals over the chord length probability distribution function (PDF) for the finite volume under examination. Current work shows that the complicated expression for the Dancoff factor as derived by Dancoff and Ginsburg can also be simplified considerably by defining a chord length PDF for the moderator region, resulting in a 1-D integral over the moderator chord length PDF. More important, this analysis

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can be extended to obtain analytical Dancoff factors for more general arrangements of fuel and moderator, including doubly heterogeneous configurations.

When neutrons are slowed by the moderator into the resonance energy range, they may be absorbed by the fuel before having their next collision with the moderator. For a single fuel lump, this resonance absorption is reduced by a certain amount due to the presence of adjacent lumps. The neighboring lumps may block the path of resonance neutrons to the fuel lump in question, reducing the probability that those resonance neutrons will reach that fuel lump before colliding with a moderator, hence reducing the "surface" resonance absorption. This effect is equivalent to increased self-shielding of the fuel and is called the "shadowing effect" in many references.[3,4] Dancoff and Ginsburg derived a formula to calculate the reduction in surface resonance absorption due to neighboring absorbers, and this reduction is known as the Dancoff-Ginsburg factor, or more commonly, the Dancoff factor.[4,5] The Dancoff factor plays an important role in calculating collision probabilities and resonance integrals for reactor fuel lattices in neutronic analysis.[6–8]

For the analysis of the Very High Temperature Gas-Cooled Reactor (VHTR), it is essential to generate accurate few-group cross sections for use in global reactor calculations.[9] In particular, one needs to account for the stochastic distribution and double heterogeneity of the TRISO fuel particle in order to obtain reasonable few-group cross sections that correctly account for resonance absorption. Although general-purpose Monte Carlo codes such as MCNP5 (Ref. 10) can analyze VHTR configurations from unit cell calculations to full-core calculations with excellent results,[11–14] the computational time is prohibitive, especially if one considers depletion and feedback effects. As a result, the conventional methodology for VHTR analysis has been based on deterministic transport methods such as MICROX-2 (Ref. 15) that depend on user-specified Dancoff factors to account for the double heterogeneity. These Dancoff factors may be calculated by Monte Carlo or semianalytical/analytical methods.[16–18] In this paper, analytical formulas for Dancoff factors based on chord length PDFs are derived. Closed-form expressions for the Dancoff factor are obtained for infinite geometry, an infinite height cylinder, a finite height cylinder, and a finite sphere. Closed-form expressions have also been obtained for finite configurations of infinite height cylinders, finite height cylinders, and spheres, but these expressions include chord length PDFs that may need to be evaluated empirically due to the complexity of analyzing finite geometries.

In the past decade, three papers[16–18] have been published on analytical calculations of Dancoff factors for pebble bed and prismatic-type reactors. These include (a) infinite medium Dancoff factors, (b) finite medium Dancoff factors for an isolated fuel pebble or fuel compact (intrapebble or intracompact), and (c) finite medium Dancoff factors for a collection of fuel pebbles or fuel compacts (interpebble and intercompact), including accounting for the coating region in a TRISO fuel particle.[18] In all of these works, the analytical expressions for the Dancoff factors are based on complex derivations that involve complicated multiple integrals over surfaces and solid angles. An alternative derivation of the Dancoff factor based on chord length PDFs is developed in this paper that replaces these complicated integrals with 1-D integrals over chord length PDFs.

In this work, a general formulation of the Dancoff factor is given, starting with Dancoff and Ginsburg's original procedure.[1] It is shown how the expression for the Dancoff factor is considerably simplified with the chord length PDF, similar to what Dirac[2] did when he studied neutron multiplication. This expression avoids the complicated double integrals over surface and solid angle that were obtained by Dancoff and Ginsburg. However, one needs to know the chord length distribution function for the moderator region, which has been addressed in previous works by the authors.[19,20]

The new expressions for the Dancoff factor are applied to the calculation of Dancoff factors for TRISO fuel kernels in several VHTR configurations. These expressions for the Dancoff factor are based on a "single-sphere" model of the TRISO fuel region, shown in Fig. 1a, which corresponds to treating the fuel region as a stochastic mixture of fuel kernels in a background medium consisting of homogenized matrix and coatings. We have also examined an alternative approach, the "dual-sphere" model, shown in Fig. 1b, which models the TRISO fuel region as a stochastic mixture of microspheres including fuel coatings. This model is attractive because the stochastic packing is constrained by the presence of the fuel coatings, which leads to a minimum separation distance between any two kernels. This in turn impacts the chord length PDF and the calculation of the Dancoff factors. However, the resultant Dancoff factors are not as accurate with the dual-sphere model as with the single-sphere model. This is currently under examination.

Analytical formulas for the Dancoff factor are obtained using the analytical chord length PDFs derived in a previous paper[19] for VHTR fuel. These closed-form expressions are used to compute Dancoff factors for both infinite medium and finite medium configurations representative of pebble bed and prismatic-type VHTRs, and excellent agreement is obtained compared to Monte Carlo benchmark results.

The remainder of this paper is organized as follows. Section II includes a derivation of the analytical formula for the Dancoff factor for an infinite medium of fuel lumps, with numerical results for VHTR configurations. Section III extends the infinite medium analysis to obtain an expression for the average Dancoff factor for a finite stochastic medium of fuel lumps, where an additional chord length PDF is defined for the finite body (e.g., a pebble) to account for neutron escape. Section IV

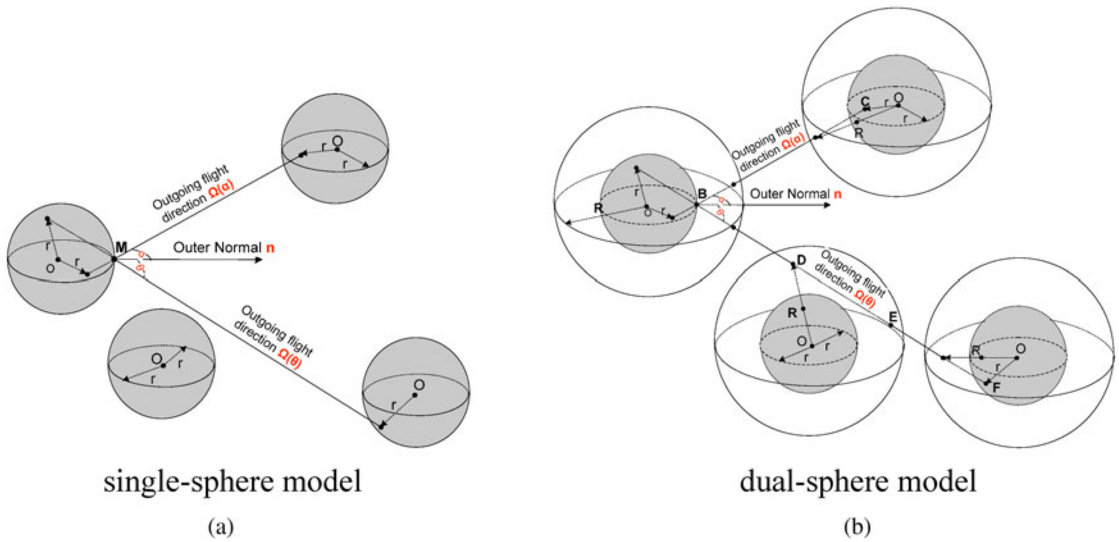

Fig. 1. VHTR fuel particle models. The dual-sphere model is defined here, but no results are reported for this model.

further extends the analysis to obtain an analytical expression for the average Dancoff factor for multiple finite media, each consisting of fuel lumps, such as a pebble bed core. This requires definition and determination of a third chord length PDF for the background region that contains the multiple finite media (e.g., the chord length PDF between pebbles in an annular core). Conclusions are given in Sec. V.

II. DANCOFF FACTOR FOR AN INFINITE MEDIUM OF FUEL LUMPS

In this section, the general formulation of the Dancoff factor for a single fuel lump in an infinite medium is given. The derivations are applicable to fuel lumps with arbitrary shapes that are uniformly dispersed in a background moderator with arbitrary packing schemes. The single-sphere model is used, and results are compared with benchmark results.

II.A. General Analytical Derivation

Consider the situation in which fuel lumps with arbitrary (nonreentrant) shapes are dispersed randomly in a background moderator. Fission neutrons are generated inside the fuel lumps and are slowed by the moderator. After several scatterings in the moderator, some of the neutrons may enter the resonance energy range and be absorbed. To model this, a uniform, isotropic source of resonance energy neutrons is assumed in the moderator.

Given the source density Q neutrons/cm$^3\cdot$s at a point $\boldsymbol{r}'$ in the moderator, the incremental uncollided flux of neutrons entering a fuel lump at the surface point $\boldsymbol{r}$ due to that source is

$$d\Psi(r' \to r) = (Qd^3r')\cdot\frac{e^{-|r-r'|/\lambda}}{4\pi|r-r'|^2} \ , \tag{1}$$

where λ is the mean free path of resonance neutrons in the moderator and it is assumed that no other fuel lump is between $\boldsymbol{r}'$ and $\boldsymbol{r}$; i.e., $\boldsymbol{r}'$ and $\boldsymbol{r}$ can see each other only through the moderator region. The incremental rate that neutrons enter the fuel lump through a small surface element dA at $\boldsymbol{r}$ is

$$\begin{aligned} dJ(r' \to r) &= d\Psi(r' \to r)\cos\theta dA \\ &= (Qd^3r')\cdot\frac{e^{-|r-r'|/\lambda}}{4\pi|r-r'|^2}\cos\theta dA \ , \end{aligned} \tag{2}$$

where θ is the angle between $\boldsymbol{r}-\boldsymbol{r}'$ and the outer normal of dA. The total rate that neutrons enter a fuel lump without colliding with the moderator is

$$\begin{aligned} J &= \iint dJ(r' \to r) \\ &= Q\int dA\cos\theta\int d^3r'\,\frac{e^{-|r-r'|/\lambda}}{4\pi|r-r'|^2} \ . \end{aligned} \tag{3}$$

Now using a spherical coordinate transformation $d^3\boldsymbol{r}' = s^2 ds d\Omega$, where $\boldsymbol{r}$ is the origin, $s = |\boldsymbol{r}-\boldsymbol{r}'|$, and $d\Omega$ is a solid angle element about direction (θ,φ), J may be expressed as

$$\begin{aligned} J &= \frac{Q}{4\pi}\int dA\int d\Omega\cos\theta\int_0^l e^{-s/\lambda}\,ds \\ &= \frac{Q\lambda}{4\pi}\int dA\int d\Omega\cos\theta(1-e^{-l/\lambda}) \ , \end{aligned} \tag{4}$$

where l is the length of the chord through the point $\boldsymbol{r}$ in the direction (θ, φ) to the surface of any other fuel lumps that can "see" the fuel lump in question. After simple manipulation, Eq. (4) can be written as

$$J = \frac{Q\lambda A}{4}\left[1 - \frac{\int dA \int d\Omega \cos\theta (e^{-l/\lambda})}{\int dA \int d\Omega \cos\theta}\right] , \tag{5}$$

where A is the total surface area of the recipient fuel lump. This is the equation Dancoff and Ginsburg obtained in their paper.[1] Now it can be easily seen that the first part in Eq. (5) is the total rate that resonance neutrons enter the fuel lump if only one fuel lump exists in the moderator. The second part is the reduction in the entering rate due to neighboring lumps. The complicated second term in the bracket is called the Dancoff factor C, which is the fractional reduction in surface resonance absorption due to the presence of the neighboring fuel regions:

$$C = \frac{\int dA \int d\Omega \cos\theta (e^{-l/\lambda})}{\int dA \int d\Omega \cos\theta} . \tag{6}$$

Next, introduce the PDF for the distribution of chord lengths in the moderator region, as Dirac did when he studied neutron multiplication for a solid of arbitrary shape.[2] According to Dirac, the chord length distribution PDF $f(l)$ is defined such that for any function $g(l)$,

$$\int f(l) g(l)\, dl = \frac{\int dA \int d\Omega \cos\theta [g(l)]}{\int dA \int d\Omega \cos\theta} . \tag{7}$$

Then Eq. (6) becomes

$$C = \int f(l) e^{-l/\lambda}\, dl . \tag{8}$$

The relative simplicity of Eq. (8) compared to Eq. (6) is clear—the four-dimensional (4-D) integral over surface and angle has been replaced by a 1-D integral over the chord length distribution. This expression for the Dancoff factor in terms of the chord length PDF was first published by Sauer[21] in 1963, but with somewhat different notation. Also, Sauer did not generalize his result to finite media, as will be discussed later in this paper. A similar result for the first flight escape probability was given by Case et al.[22] in 1953, but no mention was made of a corresponding expression for the Dancoff factor. In spite of these early references, there has been essentially no mention of the use of chord length PDFs for the calculation of Dancoff factors in stochastic media in the past 40 years, even though many papers have been published on the calculation of Dancoff factors for TRISO fuel kernels.[7,16–18,23]

As a result of Eq. (8), the calculation of the Dancoff factor is reduced to determining the chord length PDF between fuel lumps in the moderator. However, earlier work by the authors yielded an analytical expression for the chord length PDF (Ref. 19) in a stochastic medium, and this will be used to determine the Dancoff factor.

In addition to deriving Eq. (4), Dancoff performed the double integral over two fuel lump surfaces to evaluate C. However, the chord method expression for C in Eq. (8) is not only a far simpler equation, it can be evaluated easily, avoiding the substantial computational costs of evaluating the double integral, and it can be applied to stochastic media such as the distribution of TRISO fuel kernels in a VHTR configuration.

Also, Eq. (8) admits another physical explanation for the Dancoff factor. The original derivation was based on the physical fact that resonance neutrons are created uniformly in the moderator and reach a fuel lump without collision. However, Eq. (8) suggests another physical interpretation. If resonance neutrons are emitted from the surface of a fuel lump and travel through the moderator toward another fuel lump, then $f(l)dl$ is the probability that the distance to the next fuel lump is within $(l, l + dl)$ and $e^{-l/\lambda}$ is the probability that the neutron traverses the distance l without a collision. In this sense, the Dancoff factor C can be defined as an average probability that resonance neutrons escaping from a fuel lump will reach another fuel lump without experiencing any collisions with the moderator in between. This definition can be found in many references; it is equivalent to the original definition and is a consequence of the reciprocity theorem.[4,5]

II.B. Dancoff Factors for Fuel Kernels in an Infinite Medium

In this section, Eq. (8) is used to derive an analytical formula for the Dancoff factor for an infinite medium of fuel kernels in VHTR configurations. Numerical results using this expression are obtained and compared to benchmark Monte Carlo results.

II.B.1. Simplified Physical Models and Associated Mathematical Models in VHTR

In VHTR designs, either prismatic type or pebble bed type, TRISO fuel particles are randomly distributed in fuel compacts or fuel pebbles, respectively. Previous analysis has shown that homogenizing the four coating layer regions with the graphite matrix region does not affect the neutronic analysis[11] of the microsphere cell, which consists of the microsphere and its associated matrix region. Although the four coating regions and the graphite matrix region are distinct regions for the Dancoff factor calculation, these regions have identical cross sections obtained by homogenizing the materials in these five regions.

II.B.2. Analytical Derivations

The single-sphere model chord length PDF $f(l)$ was previously derived by the authors[19] and is given here:

$$f(l) = \frac{1}{\langle l \rangle} \cdot e^{-l/\langle l \rangle} \ , \quad 0 < l < \infty \ , \tag{9}$$

where

$$\langle l \rangle \equiv \frac{4r}{3} \cdot \frac{1 - frac'}{frac'}$$

is the mean chord length between two fuel kernels and $frac'$ = the ratio of total fuel kernel volume to the whole medium volume.

Substituting $f(l)$ into Eq. (8), the infinite medium Dancoff factor with the single-sphere model is obtained:

$$C^{\infty} = \int_0^{\infty} e^{-x/\lambda} \cdot \frac{1}{\langle l \rangle} \cdot e^{-l/\langle l \rangle} \cdot dl = \frac{1}{1 + \frac{1}{\lambda} \cdot \langle l \rangle}$$

$$= \frac{1}{1 + \frac{1}{\lambda} \cdot \frac{4r}{3} \cdot \frac{1 - frac'}{frac'}} \ . \tag{10}$$

It should be noted that a critical VHTR design parameter, known as the "volume packing fraction" for fuel particles in VHTR, is normally used. It is defined as $frac$ = the ratio of total microsphere fuel particle volume to the whole medium volume and is related to $frac'$ by $frac' = frac \cdot (r/R)^3$. Unless specified, the volume packing fraction mentioned in this paper refers to $frac$.

II.B.3. Early Calculations of Dancoff Factors for Infinite Stochastic Media

The earliest papers that discussed the evaluation of the Dancoff factor for an infinite medium with grain structure are those by Lane et al.[24] and Nordheim[25] in 1962. These papers gave a result similar to Eq. (10), but with a different mean chord length $\langle l \rangle = 4r/3 \cdot (1/frac')$. Since they used a classical atomic structure model to obtain the mean chord length, which assumes that the fuel kernels are infinitesimal, the mean chord length was overestimated. Their derivations miss the factor $(1 - frac')$ in the numerator, which accounts for the finite size of the grains, hence leading to an incorrect homogeneous limit.[25,26] Although Nordheim had pointed out this homogeneous limit inconsistency, he did not give a solution to solve it. Lewis and Connolly[26] also noted this and used an alternative result due to Bell[27] to obtain the correct homogeneous limit. It is clear that this inconsistency was due to the incorrect evaluation of the mean chord length in the moderator. This has been corrected in our expression for the infinite medium Dancoff factor in Eq. (10), which yields the correct homogeneous limit for the escape probability.

Also, Lane et al.[24] suggested a method to account for the coating region by changing the lower integral limit from 0 to $2d$ in Eq. (10). However, this treatment gave poor results since the chord length PDF should also be changed if a coating region is added to the fuel kernel, but this was not done by Lane. One needs to simultaneously change the PDF to account for the coating region as well as modify the lower limit of the integral. This is what the dual-sphere model accomplishes and is the motivation for continuing our examination of this model.

II.C. Monte Carlo Benchmark Calculation

Equation (10) is the analytical formula for infinite medium Dancoff factors for the single-sphere model and is based on Eq. (8). To verify the accuracy of this formula, Monte Carlo benchmark simulations were performed to calculate the following quantities:

1. Dancoff factors for fuel kernels in an infinite stochastic medium as a function of volume packing fraction. These results are compared directly with the predictions from Eq. (10).

2. empirical chord length PDFs between fuel kernels in an infinite stochastic medium. These empirical PDFs are used for a self-consistent verification of Eq. (8).

The benchmark Monte Carlo simulations employ conventional ray-tracing methods to directly track neutron trajectories in explicit stochastic mixtures of kernels. The stochastic mixtures are generated with a modified version of the fast random sequential addition (RSA) algorithm[28,29] to generate multiple realizations of the stochastic media at different volume packing fractions.

A Monte Carlo code was written to calculate Dancoff factors of fuel kernels in an infinite medium. Depending on the packing fraction, approximately 5 million to 10 million TRISO fuel particles are dispersed randomly in a graphite background region. Neutrons are emitted uniformly from the surface of fuel kernels with a cosine current distribution. The number of neutrons that successfully reach another fuel kernel without having a collision in the graphite region is tallied, and the Dancoff factor is computed as the ratio of this number to the total number of neutrons emitted. A total of 100 physical realizations of the stochastic medium are performed, and 10 million neutrons are emitted per realization. The final Dancoff factor is ensemble averaged over the 100 realizations.

TABLE I

Values of Parameters

Parameter	Value	Unit
r	0.0175	cm
R	0.0390	cm
$2d = 2(R - r)$	0.043	cm
$1/\lambda$	0.4137	cm^{-1}

Geometry and composition quantities used for the Dancoff factor calculations and Monte Carlo simulations are given in Table I. The geometry data correspond to the NGNP Point Design,[30] and cross-section data are from the Brookhaven National Laboratory website[31] evaluated at the 6.67-eV resonance of ^{238}U.

To compute the actual chord length PDF between fuel kernels at different volume packing fractions, the same geometry and source distribution are used as in the benchmark calculation for the Dancoff factors, except the number of fuel particles is increased to 10 million to 20 million, and 500 000 neutrons are emitted for each of the 100 realizations. Figure 2 shows the results as a function of volume packing fraction from 5.76% (typical of a pebble bed reactor) to 28.92% (typical of a prismatic reactor). All the PDFs are nonzero beginning with the chord length $2d = 2(R - r) = 0.043$ cm, as expected. The PDFs quickly increase to a peak value and decay exponentially for long chord lengths. The log-linear scale plot verifies that the chord length PDF is well approximated by an exponential distribution and may be used for the calculation of Dancoff factors for this range of packing fractions. Similar results have been obtained by other researchers.[32]

II.D. Numerical Results

Table II compares Dancoff factors predicted with Eq. (10) with the benchmark Monte Carlo results. The results are excellent, within 1.1% for all volume packing fractions, with somewhat better results for higher packing fractions. Interestingly, all the predicted Dancoff factors are higher than the benchmark results.

The introduction of the chord length PDF makes the Dancoff factor calculation succinct and mathematically simple and yields excellent results for the single-sphere model. Equation (8) is a fundamental equation for the Dancoff factor not only in an infinite medium but also in a finite medium, as will be seen in later sections. Because of its importance, an additional comparison is made to verify its self-consistency; i.e., if the exact chord length PDF is used in Eq. (8), the resultant Dancoff factor should be exact. This provides additional assurance of the methodology that employs Eq. (8). To do this, Dancoff factors are calculated using an empirical chord length PDF as shown in Fig. 2 and compared with the Monte Carlo benchmark results. As discussed earlier, the empirical chord length PDF for the fuel kernels was based on an exhaustive set of Monte Carlo simulations in random mixtures of fuel particles and may be considered close to exact. Therefore, using this empirical PDF in Eq. (8) essentially constitutes a Monte Carlo benchmark calculation for the Dancoff factor and should yield the same results. Table III compares the Dancoff factors calculated with the empirical PDF in Eq. (8) with the Monte Carlo benchmark results, and the agreement is within 0.3%, with no observable trend with packing factor. We conclude that our methodology for estimating Dancoff factors based on chord length PDFs is accurate and self-consistent.

III. DANCOFF FACTOR FOR A FINITE MEDIUM OF FUEL LUMPS

In the previous section, analytical formulas of Dancoff factors for an infinite stochastic medium were shown to be accurate for packing factors representative of prismatic and pebble bed VHTRs. Although this is an

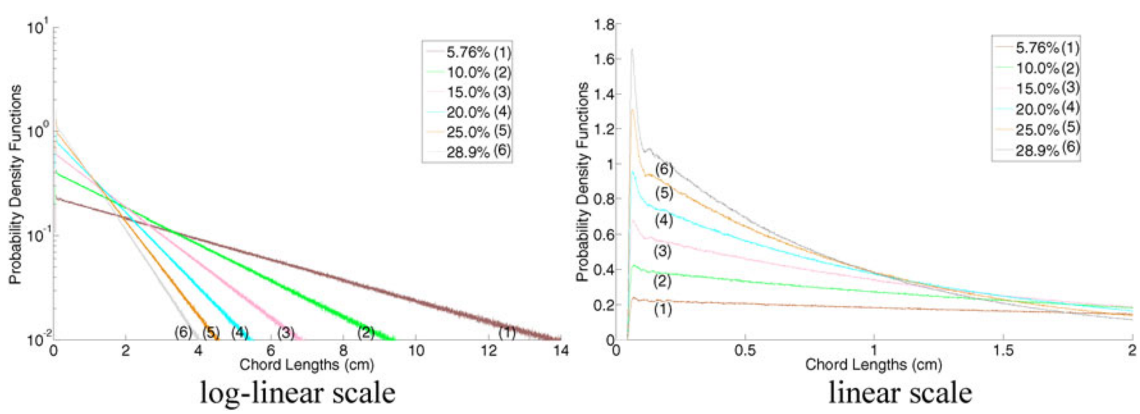

Fig. 2. Chord length PDFs between two fuel kernels.

TABLE II

Analytical Results with Single-Sphere Model Compared to Benchmark Results

Volume Packing Fraction	Analytical Formula Result C^{∞} [Eq. (10)]	Monte Carlo Benchmark Result $C^{B,\infty}$ (1σ)	Difference $C^{\infty} - C^{B,\infty}$	Relative Error $(C^{\infty} - C^{B,\infty})/C^{B,\infty}$ (%)
0.0576	0.3515	0.3477 (0.0002)	0.0038	1.09
0.10	0.4857	0.4820 (0.0002)	0.0037	0.77
0.15	0.5873	0.5841 (0.0002)	0.0032	0.55
0.20	0.6559	0.6534 (0.0001)	0.0025	0.38
0.25	0.7054	0.7029 (0.0001)	0.0025	0.35
0.2892	0.7353	0.7331 (0.0001)	0.0021	0.30

TABLE III

Analytical Results from Eq. (8) Using Simulated PDF Compared to Benchmark Results

Volume Packing Fraction	Analytical Formula Result $C^{A,\infty}$ [Eq. (8)]	Monte Carlo Benchmark Result $C^{B,\infty}$ (1σ)	Difference $C^{A,\infty} - C^{B,\infty}$	Relative Error $(C^{A,\infty} - C^{B,\infty})/C^{B,\infty}$ (%)
0.0576	0.3471	0.3477 (0.0002)	−0.0006	−0.17
0.10	0.4810	0.4820 (0.0002)	−0.0010	−0.21
0.15	0.5827	0.5841 (0.0002)	−0.0014	−0.24
0.20	0.6516	0.6534 (0.0001)	−0.0018	−0.27
0.25	0.7014	0.7029 (0.0001)	−0.0015	−0.21
0.2892	0.7316	0.7331 (0.0001)	−0.0015	−0.20

interesting theoretical result, the practical impact is not significant when one is faced with analyzing a finite medium, such as a cylindrical fuel compact or spherical fuel pebble that is filled with TRISO fuel particles. In this case the average Dancoff factor for a TRISO fuel kernel in the finite medium is needed to determine space-dependent few-group cross sections.

III.A. General Derivations

The key observation is that Eq. (8) can be used to calculate the Dancoff factor for a single fuel lump in a finite medium if the upper limit is set to a finite value related to the position of the fuel lump in the medium and the direction of neutron travel, as shown in Fig. 3. Moreover, this is essentially the same situation that is encountered when computing the average escape probability for a point source in a finite medium, which was solved using Dirac's chord method. A similar analysis leads to another analytical formula for the average Dancoff factor in a finite stochastic medium of fuel lumps, as discussed below.

To obtain the average Dancoff factor for all the fuel lumps in the volume, the integral in Eq. (8) needs to be performed over the surface area of each fuel lump and over the entire finite medium, accounting for every fuel lump. This would be a cumbersome integration. However, if the chord method is employed to perform the calculation as was done in Sec. II, the calculation is much

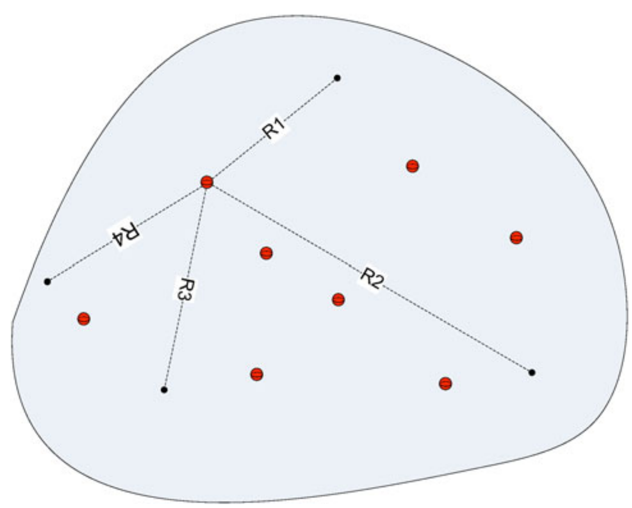

Fig. 3. Fuel lumps in finite volume.

simpler. Here one takes advantage of the fact that the fuel lumps are randomly distributed in the finite medium. The physical process to be simulated is summarized as follows: A source of resonance energy neutrons is uniformly distributed on the surface of a fuel lump, and a resonance neutron is emitted outward with a cosine current angular distribution. The probability that a neutron enters another fuel lump without colliding with the intervening moderator is calculated and averaged over the surface neutron emissions. This average probability is only for a specific fuel lump at a specific location inside the solid body. Since the fuel lump could be anywhere in the volume, we need to average this to account for fuel lumps at different locations in the volume. However, it should be noted that a uniform distribution of fuel lumps with uniform cosine current surface sources is equivalent to a uniform, isotropic volumetric source in the moderator because the surface source can appear anywhere in the volume with equal probability. At this point, the mathematical derivation becomes equivalent to that performed by Case et al.[22] and Bell and Glasstone[5] when they calculated escape probabilities for finite volumes. This derivation is given below.

In the moderator with a uniform isotropic volumetric source, a resonance energy neutron is generated in $d\Omega dV$ with probability $d\Omega dV/4\pi V$, traveling in the direction $d\Omega$ about Ω. Since fuel lumps are randomly located in the moderator, this neutron may be regarded as escaping the fuel lump with a cosine current distribution. The maximum distance along the neutron trajectory to enter another fuel lump within the finite medium is denoted and is determined by the distance to the boundary of the finite medium along the chord determined by $(\boldsymbol{r}, \Omega)$. Strictly speaking, this distance should be reduced by the average chord length in the fuel lump since a fuel lump cannot overlap the outer boundary, but this has not been done in this work. Since the probability of traversing distance l without a collision in the moderator is given by $e^{-l/\lambda}$, the following expression for the Dancoff factor is obtained:

$$C(l) = \int_{\mathrm{min\_}d}^{l} f(l')e^{-l'/\lambda}\, dl' \ , \qquad (11)$$

where min_d is a problem-dependent lower limit (such as the minimum distance between two TRISO fuel kernels due to the coatings) and l is the distance to the boundary of the finite medium along the neutron trajectory. This is the Dancoff factor for a specific neutron emitted randomly from the surface of a random fuel lump within the finite medium. The probability of choosing this specific neutron is $d\Omega dV/(4\pi V)$; hence, the average Dancoff factor over all fuel lumps in the volume is given by

$$C^{intra} = \iint \frac{1}{4\pi V}\, C(l)\, dV\, d\Omega \ . \qquad (12)$$

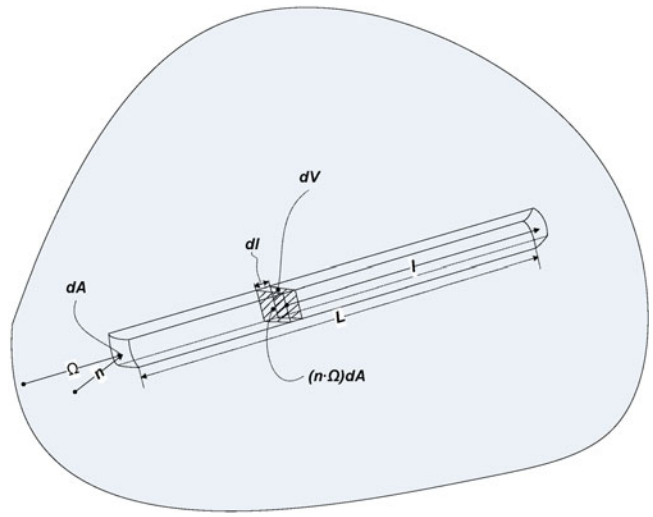

Fig. 4. Volume element division.

Note that l is a function of $(\boldsymbol{r}, \Omega)$ and min_d is a constant depending on the TRISO fuel dimensions. Equation (12) is a complicated expression for the average Dancoff factor due to the implicit dependence of l on $(\boldsymbol{r}, \Omega)$. From Fig. 4, $dV = (n \cdot \Omega) dA dl$, so Eq. (12) becomes

$$C^{intra} = \frac{1}{4\pi V} \int dA \int d\Omega \cos\theta \int_{\mathrm{min\_}d}^{L} C(l)\, dl \ , \qquad (13)$$

where min_d has the same meaning as in Eq. (11) and L is now the maximum distance to the boundary. Equation (13) is just as complex to evaluate as Eq. (12), but it is in a form where the chord method can again be used to advantage. Define

$$G(L) = \int_{\mathrm{min\_}d}^{L} C(l)\, dl$$

and use

$$\int dA \int d\Omega \cos\theta = \pi A \ ,$$

where A is the total surface area of the finite medium, yielding

$$C^{intra} = \frac{\pi A}{4\pi V} \frac{\int dA \int d\Omega \cos\theta G(L)}{\int dA \int d\Omega \cos\theta} = \frac{1}{\langle L \rangle} \frac{\int dA \int d\Omega \cos\theta G(L)}{\int dA \int d\Omega \cos\theta} \ , \qquad (14)$$

where $\langle L \rangle = 4V/A$, the mean chord length for the finite medium. As was done earlier with Eq. (7) in Sec. II.A, introduce the chord distribution function $F(L)$ for the finite medium, leading to the following simple expression for C^{intra}:

$$C^{intra} = \frac{1}{\langle L \rangle} \int F(L) G(L)\, dL \ , \tag{15}$$

or a more explicit form,

$$C^{intra} = \frac{1}{\langle L \rangle} \int dL F(L) \int_{\min\_d}^{L} dl \int_{\min\_d}^{l} dl' f(l') e^{-l'/\lambda} \ . \tag{16}$$

The complicated expression for the average Dancoff factor in a finite medium of fuel lumps, which consists of multiple 4-D integrals over surface and angle domains, is reduced to a straightforward multiple integral by using two chord length distribution functions: (a) the PDF $f(l)$ for the distribution of chord lengths between fuel lumps and (b) the PDF $F(L)$ for the distribution of chord lengths inside the finite medium.

III.B. Dancoff Factor Calculations for a Finite Medium

In this section, average Dancoff factors for finite VHTR geometries, including fuel compacts and fuel pebbles, will be calculated. For a single-sphere model, substituting $f(l)$ from Eq. (9) into Eq. (16), the intravolume Dancoff factor with the single-sphere model is obtained:

$$C^{intra} = \frac{1}{\langle L \rangle} \int dL F(L) \int_0^L dl \int_0^l dl' \frac{1}{\langle l \rangle} e^{-l'/\langle l \rangle} e^{-l'/\lambda} \ , \tag{17}$$

where we have set $\min\_d = 0$ due to the single-sphere model assumption.

Now define an effective cross section Σ^*:

$$\Sigma^* = \frac{1}{\lambda^*} = \frac{1}{\lambda} + \frac{1}{\langle l \rangle} \ , \tag{18}$$

which leads to the following expression after several straightforward manipulations:

$$C^{intra} = \frac{\lambda^*}{\langle l \rangle} \left[1 - \frac{\lambda^*}{\langle L \rangle} \int (1 - e^{-L/\lambda^*}) F(L)\, dL \right] \ . \tag{19}$$

It is easy to see from Eq. (10) that $\lambda^*/\langle l \rangle$ is just C^∞. It is also found with more careful observation that the second term in the bracket is just the first flight escape probability for a finite medium with a total cross section Σ^* (see page 22 in Ref. 22). The effective cross section Σ^* is identical to the modified total cross section for a two-region lattice in equivalence theory when the "escape" cross section $1/\langle l \rangle$ is added to the total cross section.[5]

This leads to a very concise result:

$$C^{intra} = C^\infty [1 - P^*_{esc}] \ . \tag{20}$$

Thus, the average intramedium Dancoff factor is the product of the infinite medium Dancoff factor and the first flight collision probability for the finite medium. Bende et al.[16] and Talamo[17] also found a similar expression for C^{intra} for fuel pebbles and fuel compacts, respectively. However, their analyses are somewhat more complicated due to a different definition of Σ^* and the complicated integral expression for P^*_{esc}.

The introduction of the chord length PDF $F(L)$ in the expression of C^{intra} in Eq. (19) makes the formula more flexible, in that it can handle different finite media with arbitrary (nonreentrant) shapes filled with randomly distributed fuel particles. Next, as an application to VHTR analysis, Eqs. (19) and (20) are used to derive intrapebble and intracompact Dancoff factors in terms of basic geometry parameters.

A fuel pebble is composed of two concentric spheres: an inner sphere with a typical radius $R_1 = 2.5$ cm and an outer sphere with a typical radius $R_2 = 3.0$ cm. The inner sphere is the fuel zone with fuel particles randomly distributed within it, and the outer spherical shell is graphite.

The chord length PDF for a sphere of radius R_1 is well known (see Appendix of Ref. 33):

$$F(L) = \frac{L}{2R_1^2} \ , \quad 0 < L < 2R_1 \ . \tag{21}$$

Substituting this into Eq. (19), the following expression for P^*_{esc} is found after several algebraic manipulations:

$$P^*_{esc} = \frac{3}{4}\left(\frac{\lambda^*}{R_1}\right) + \frac{3}{4}\left(\frac{\lambda^*}{R_1}\right)^2 e^{-2(R_1/\lambda^*)} - \frac{3}{8}\left(\frac{\lambda^*}{R_1}\right)^3 [1 - e^{-2(R_1/\lambda^*)}] \ . \tag{22}$$

This leads to the following equation for the Dancoff factor:

$$C^{intra} = C^\infty \left\{ 1 - \frac{3}{4}\left(\frac{\lambda^*}{R_1}\right) - \frac{3}{4}\left(\frac{\lambda^*}{R_1}\right)^2 e^{-2(R_1/\lambda^*)} + \frac{3}{8}\left(\frac{\lambda^*}{R_1}\right)^3 [1 - e^{-2(R_1/\lambda^*)}] \right\} \ . \tag{23}$$

As verification, when $R_1 \to \infty$, $C^{intra} \to C^\infty$.

In the prismatic VHTR, the fuel compacts are arranged in cylindrical columns of 793-cm height. Although these are essentially infinite cylinders, an expression for C^{intra} will be derived for both infinite and finite cylinder cases.

For an infinite cylinder region, the chord length PDF was derived by Case et al.[22]:

$$F(L) = \frac{16R_c^2}{\pi L^3}\int_0^{x_0}\frac{x^4\,dx}{\sqrt{(L^2/4R_c^2)-x^2}\sqrt{1-x^2}} \ ,$$

$$x_0 = \begin{cases} 1 & L > 2R_c \\ \dfrac{L}{2R_c} & L < 2R_c \ . \end{cases} \qquad (24)$$

Using the above PDF and results from Corngold[34] for P_{esc}^* in terms of Bessel functions, one finds

$$P_{esc}^* = \frac{2}{3}\{2t^2[I_0(t)K_0(t) + I_1(t)K_1(t)] + t[I_0(t)K_1(t) - I_1(t)K_0(t) - 2] + I_1(t)K_1(t)\} \ , \qquad (25)$$

where $t = R_c/\lambda^*$. Substituting this into Eq. (20), the following result is obtained:

$$C^{intra} = C^{\infty}\left\langle 1 - \frac{2}{3}\{2t^2[I_0(t)K_0(t) + I_1(t)K_1(t)] + t[I_0(t)K_1(t) - I_1(t)K_0(t) - 2] + I_1(t)K_1(t)\}\right\rangle \ . \qquad (26)$$

As verification, when $R_c \to \infty$, $t \to \infty$, so $P_{esc}^* = (1/2) \times (1/t) - (3/32) \times (1/t^3) + \cdots \to 0$; hence, $C^{intra} \to C^{\infty}$.

For a finite cylinder, the analytical derivation of $F(L)$ has been studied by many researchers.[35,36] Formulas for right circular cylinders and general cylinders have been found, but they are given in complicated forms. However, by using some simple mathematical transformations, researchers[37,38] have derived an expression for the escape probability P_{esc}^* for a finite cylinder, which can be used directly to obtain C^{intra}. According to Marleau et al.,[37] the following results for the escape probabilities have been obtained:

$$P_{esc,\,t\&b}^* = \frac{\lambda^*}{H}\left[E_3(0) - E_3\left(\frac{H}{\lambda^*}\right)\right] - \frac{4}{\pi D^2}\frac{\lambda^*}{H} \times \int_0^D \left\{E_3\left(\frac{t}{\lambda^*}\right) - E_3\left[\frac{(t^2+H^2)^{1/2}}{\lambda^*}\right]\right\} \times (D^2 - t^2)^{1/2}\,dt \ , \qquad (27)$$

$$P_{esc,cyl}^* = \frac{4}{\pi D^2 H}\int_0^D t^2\,dt \int_0^H (H-u) \times \frac{\exp\left(-\dfrac{(t^2+u^2)^{1/2}}{\lambda^*}\right)(D^2-t^2)^{1/2}}{(t^2+u^2)^{3/2}}\,du \ , \qquad (28)$$

$$P_{esc,cyl}^* = \frac{4}{\pi D^2 H}\int_0^D t^2\,dt \int_0^H (H-u) \times \frac{\exp\left(-\dfrac{(t^2+u^2)^{1/2}}{\lambda^*}\right)(D^2-t^2)^{1/2} - D}{(t^2+u^2)^{3/2}}\,du + \frac{4}{\pi D^2 H}\frac{D}{2} \times \left\{H^2 \ln\left[\frac{D + (D^2+H^2)^{1/2}}{H}\right] + D(D^2+H^2)^{1/2} - D^2\right\} \ , \qquad (29)$$

where $P_{esc,\,t\&b}^*$ is the axial escape probability across the top and bottom surfaces, $P_{esc,cyl}^*$ is the radial escape probability across the cylindrical surface, and $D = 2R_c$. Adding these two escape probabilities yields the total escape probability for a finite cylinder:

$$P_{esc}^* = P_{esc,\,t\&b}^* + P_{esc,cyl}^* \ . \qquad (30)$$

So the final result is

$$C^{intra} = C^{\infty}(1 - P_{esc}^*) \ . \qquad (31)$$

III.C. Monte Carlo Benchmark Calculation

Monte Carlo benchmark simulations are performed to verify the accuracy of the analytical formulas for average Dancoff factors of fuel kernels in fuel pebbles and fuel compacts. The fuel compact and fuel pebble dimensions correspond to the NGNP Point Design.[30]

For Dancoff factors in a pebble bed reactor, as shown in Fig. 5, the fuel pebble is modeled as two concentric spheres. The inner sphere has a radius $R_1 = 2.5$ cm and represents the fuel zone filled with a stochastic distribution of TRISO fuel particles. The outer spherical shell represents the moderator region with outer radius R_2 that is typically 3.0 cm but may change depending on the fuel/moderator ratio in a pebble bed reactor. This geometry is used to calculate both intrapebble and interpebble Dancoff factors. The latter will be discussed in Sec. IV.

To estimate intrapebble Dancoff factors, from 5000 to 70 000 fuel particles are packed inside the fuel zone

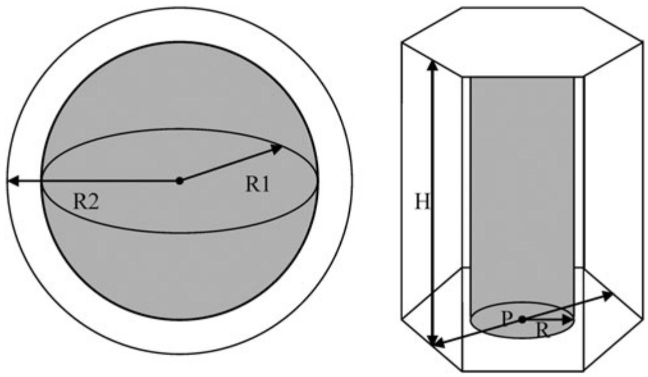

Fig. 5. Fuel pebble and fuel compact cells in Monte Carlo simulation.

using the RSA algorithm, corresponding to a range of packing fractions from 2% to 25%. Neutrons are emitted uniformly from the surface of the fuel kernels with a cosine current distribution. The number of neutrons that successfully reach another fuel kernel without having a collision is tallied, and the Dancoff factor is the ratio of this number to the total number of neutrons emitted.

To estimate Dancoff factors in a prismatic-type reactor, a hexagonal lattice structure is set up to model a fuel compact cell (Fig. 5). The fuel compact has radius $R_c = 0.6225$ cm and the outer hexagonal graphite region has a flat-to-flat (one side to opposite side of hexagon) distance $P = 2.196$ cm. Both infinite cylinders and finite cylinders are modeled.

For the infinite cylinder, a height $H = 200$ cm was chosen with reflecting boundaries on the top and bottom. Fuel particles are randomly packed inside the cylinder with packing fractions from 2% to 28.92%. There are 100 realizations with 10 million neutrons emitted per realization, similar to the pebble bed analysis.

For the finite cylinder, the packing fraction is fixed at 28.92% and H is adjusted from 2 to 100 cm for the intracompact Dancoff factor computations. The other settings are the same as for the infinite cylinder case.

III.D. Numerical Results

In this section, numerical results for average Dancoff factors for fuel kernels in fuel pebbles and fuel compacts from the analytical formulas given earlier are compared with the Monte Carlo benchmark results.

For intrapebble Dancoff factors, results are compared as a function of volume packing fraction ranging from 2% to 25%. Results for analytical results using the single-sphere model are plotted in Fig. 6, along with the Monte Carlo benchmark results. Figure 6 shows that the single-sphere model gives excellent results over the range of volume packing fractions, especially at higher packing fractions.

Figures 7 and 8 compare benchmark Monte Carlo results with the results of the analytical expressions for average Dancoff factors in fuel compacts modeled as cylinders, including both infinite and finite height compacts at packing fractions ranging from 2% to 28.92%.

For the infinite cylinder, it is interesting to see that for all packing fractions, the single-sphere model overestimates the Dancoff factors, compared to the benchmark Monte Carlo results. This was also observed for the infinite medium Dancoff factors. The agreement is better at low packing fraction than high packing fraction but all in the acceptable range. In general, the single-sphere model

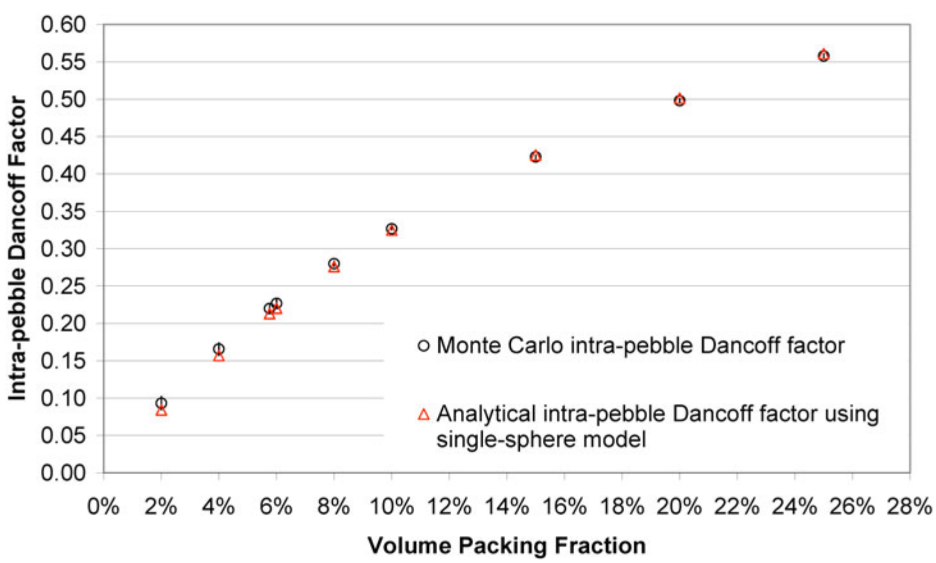

Fig. 6. Comparison between analytical and Monte Carlo average intrapebble Dancoff factors.

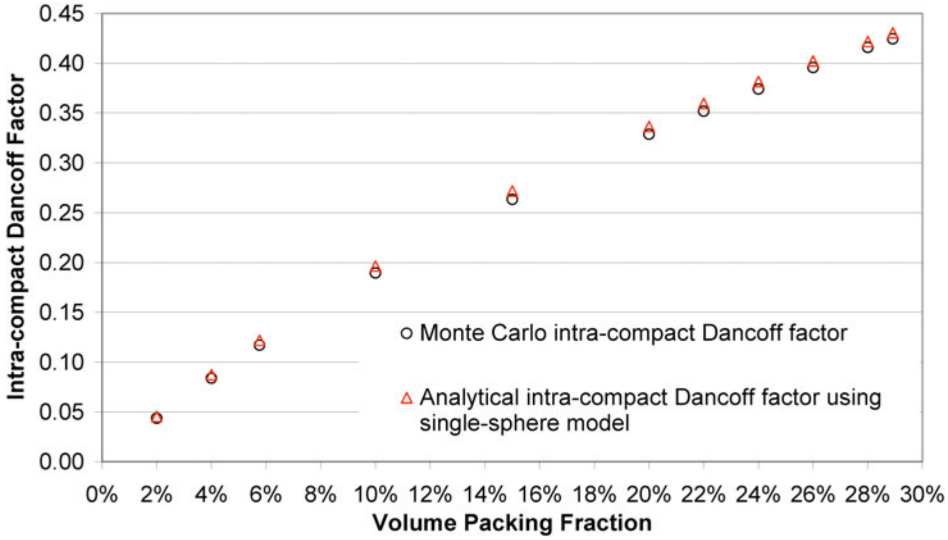

Fig. 7. Comparison between analytical and Monte Carlo average intracompact Dancoff factors for infinite height compacts.

gives reasonable results for the entire range of packing fractions.

For the finite cylinder, the volume packing fraction of fuel particles is fixed at the nominal value 28.92% within the compact and the height of the hexagonal cell (H) is varied from $2 \times D$ to $100 \times D$, where D is the diameter of the fuel compact. Figure 8 shows the results for the intracompact Dancoff factors as a function of the compact height. Similar to the Dancoff factor for an infinite medium, the single-sphere model overestimates the results over the range of heights but overall gives very good acceptable results.

IV. AVERAGE DANCOFF FACTORS FOR MULTIPLE FINITE REGIONS

The previous sections derived analytical expressions for average Dancoff factors for infinite and finite regions comprised of a stochastic mixture of fuel lumps, characteristic of an infinite medium or a finite region of TRISO fuel such as a fuel compact or fuel pebble, respectively. This section extends the analysis to include multiple finite regions, such as a prismatic core comprised of many fuel compacts or a pebble bed core consisting of many fuel pebbles. When the Dancoff factor is

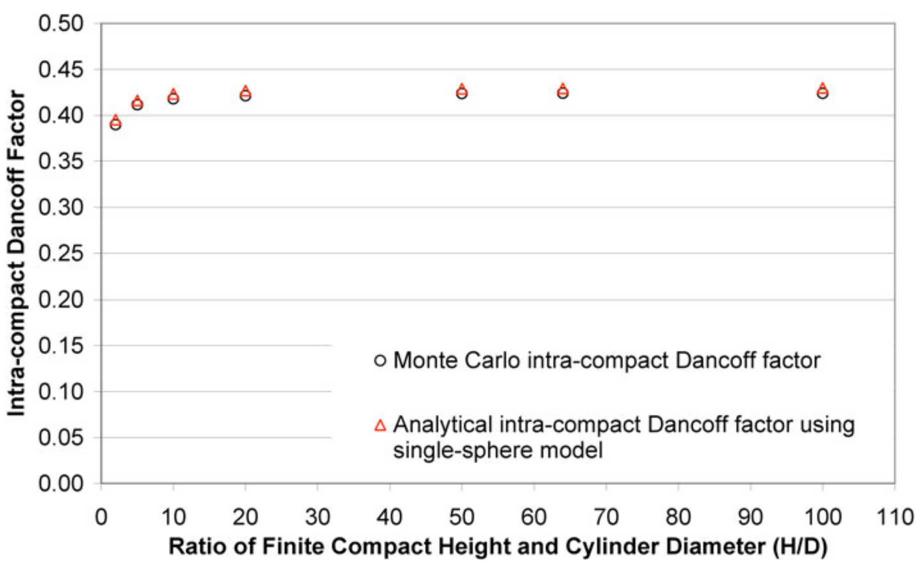

Fig. 8. Comparisons between analytical and Monte Carlo average intracompact Dancoff factors for finite height compacts.

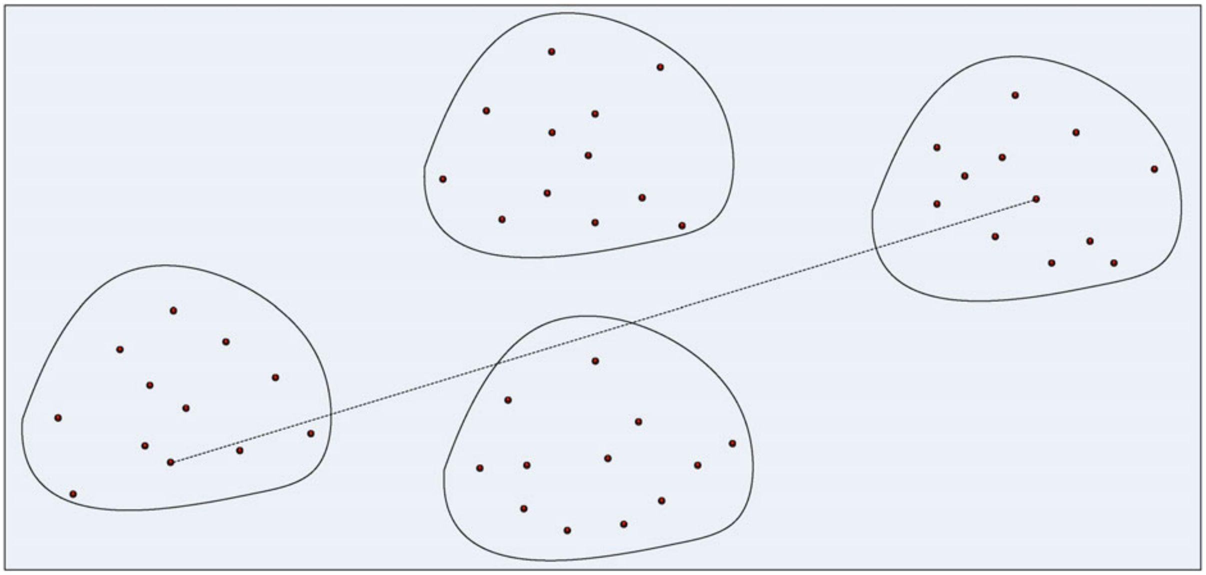

Fig. 9. Distribution of finite volumes containing fuel lumps.

calculated for a fuel volume, one needs to account not only for the intravolume contribution discussed in the previous section but also for the intervolume contribution since a neutron might exit a finite volume and stream to another finite volume without a collision. Figure 9 shows this general configuration, where the average Dancoff factor will depend on the intervolume contributions.

IV.A. General Analytical Derivations

The intervolume Dancoff factor is defined as the probability that a neutron escaping from a fuel lump in a finite volume enters another fuel lump in a different finite volume. This probability can be expressed in terms of several basic probabilities:

P_1 = average probability that a neutron escaping from a fuel lump in a finite volume escapes the volume without entering another fuel lump or colliding with the moderator. This is identical to the first flight escape probability P^*_{esc}.

P_2 = average probability that a neutron escaping from a finite volume enters another finite volume without a collision.

P_{tr} = average probability that a neutron incident on a finite volume traverses it without entering any fuel lump or having a collision with the moderator.

P_3 = average probability that a neutron incident on a finite volume enters a fuel lump within that volume.

Assuming that the finite volumes are uniformly distributed in an infinite background medium, and assuming that average probabilities can be used for the actual probabilities describing specific neutron trajectories, the intervolume Dancoff factor can be expressed as

$$\begin{aligned} C^{inter} &= P_1 P_2 P_3 + P_1 P_2 P_{tr} P_2 P_3 + P_1 (P_2 P_{tr})^2 P_2 P_3 \\ &\quad + P_1 (P_2 P_{tr})^3 P_2 P_3 + \cdots \\ &= P_1 P_2 P_3 \sum_{i=0}^{\infty} (P_2 P_{tr})^i \ . \end{aligned} \tag{32}$$

The series can be summed, yielding

$$C^{inter} = P_1 P_2 P_3 \frac{1}{1 - P_2 P_{tr}} \ . \tag{33}$$

The chord method can be used to derive expressions for these probabilities.

To obtain an expression for P_1, one proceeds in a manner similar to the derivation of C^{intra} that led to Eq. (16). The average probability that a neutron in $d\Omega dV$ escapes a finite volume along Ω without entering another fuel lump or colliding with the moderator is given by (see Fig. 4)

$$P_1(l) = \frac{d\Omega dV}{4\pi V} e^{-l/\lambda} \int_l^{\infty} f(l')\, dl' \ , \tag{34}$$

where l is the distance along the neutron trajectory from $d\Omega dV$ to the exiting point of the volume. Integrating over $d\Omega dV$, one obtains

$$P_1 = \iint \frac{d\Omega dV}{4\pi V} e^{-l/\lambda} \int_l^{\infty} f(l')\,dl' = \frac{\int dA \int d\Omega \cos\theta}{4\pi V} \frac{\int dA \int d\Omega \cos\theta \int_{\min\_d}^{L} dle^{-l/\lambda} \int_l^{\infty} f(l')\,dl'}{\int dA \int d\Omega \cos\theta} \quad . \tag{35}$$

Using the chord length distribution function in a finite body $F(L)$ that was introduced in the previous section in Eq. (15), the final expression for P_1 is obtained:

$$P_1 = \frac{1}{\langle L\rangle} \iint \frac{d\Omega dV}{4\pi V} e^{-l/\lambda} \int_l^{\infty} f(l')\,dl' = \frac{1}{\langle L\rangle} \int dLF(L) \int_{\min\_d}^{L} dle^{-l/\lambda} \int_l^{\infty} f(l')\,dl' \quad . \tag{36}$$

Assuming $H(S)$ as the chord length PDF between two volumes in an infinite background medium of volumes, P_2 is readily found to be

$$P_2 = \int H(S) e^{-S/\lambda}\,dS \quad , \tag{37}$$

where $H(S)$ is assumed to be known, perhaps empirically, for the given geometry.

To determine P_{tr}, begin with the probability that a neutron incident on a finite volume through a small element $dAd\Omega$ along Ω crosses the finite volume without hitting any fuel lump or colliding with the moderator:

$$P_{tr}(L) = e^{-L/\lambda}\left(1 - \int_0^L f(l')\,dl'\right) = e^{-L/\lambda} \int_L^{\infty} f(l')\,dl' \quad , \tag{38}$$

where L is the chord length from $dAd\Omega$ to the exit point along Ω. The first term is the probability that no collisions with the moderator happen, and the second term is the probability that no fuel lumps are hit. Integrating $P_{tr}(L)$ over all possible chords in the finite volume assuming that incident neutrons have a cosine current distribution, an expression for P_{tr} is found:

$$P_{tr} = \frac{\iint dAd\Omega \cos\theta P_{tr}(L)}{\iint dAd\Omega \cos\theta} = \int dLF(L) e^{-L/\lambda} \int_L^{\infty} f(l')\,dl' \quad . \tag{39}$$

The probability P_3 may be viewed as the Dancoff factor for a neutron incident on the boundary of the finite volume. Therefore, using our previous results, the average probability that a neutron entering a finite volume will enter a fuel region within that volume without colliding with the moderator is

$$P_3(L) = \int_{\min\_d}^{L} f(l) e^{-l/\lambda}\,dl \quad , \tag{40}$$

where L is the chord length from the incident point to the exit point along Ω. This is then averaged over all possible chords in the finite volume:

$$P_3 = \frac{\iint dAd\Omega \cos\theta P_3(L)}{\iint dAd\Omega \cos\theta} = \int dLF(L) \int_{\min\_d}^{L} f(l) e^{-l/\lambda}\,dl \quad . \tag{41}$$

Now all of the terms P_1, P_2, P_3, and P_{tr} have been expressed in terms of chord length distribution functions. Substituting Eqs. (36), (37), (39), and (41) into Eq. (33), one obtains the final analytical formula for C^{inter}:

$$C^{inter} = \frac{\left(\frac{1}{\langle L\rangle} \int dLF(L) \int_{\min\_d}^{L} dle^{-l/\lambda} \int_l^{\infty} f(l')\,dl'\right) \cdot \left(\int H(S) e^{-S/\lambda}\,dS\right) \cdot \left(\int dLF(L) \int_{\min\_d}^{L} f(l) e^{-l/\lambda}\,dl\right)}{1 - \left(\int H(S) e^{-S/\lambda}\,dS\right) \cdot \left(\int dLF(L) e^{-L/\lambda} \int_L^{\infty} f(l')dl'\right)} \quad . \tag{42}$$

Equation (42) is an analytical expression for the Dancoff factor in a collection of multiple finite stochastic regions. It depends on three chord length PDFs, although one of these, $H(S)$, may need to be determined empirically. This expression, consisting of three-dimensional (or lower dimensionality) integrals over chord length PDFs, represents a considerable reduction in complexity from what in principle would be a 12-dimensional integral: multiple two-dimensional integrals both in surface and solid angle for each of three imbedded geometries—TRISO particles, finite volumes of TRISO particles (e.g., compacts or pebbles), and the outer region consisting of multiple finite volumes (e.g., pebble bed or prismatic reactor core).

IV.B. Dancoff Factor Expressions for Finite VHTR Configurations

The results of the previous section will be applied to both prismatic and pebble bed VHTR configurations with multiple fuel compacts and multiple fuel pebbles, respectively. Substituting Eq. (9) into Eqs. (36), (39), and (41) and using the effective cross section Σ^* defined by Eq. (18), the following expressions are obtained:

$$P_1 = \frac{1}{\langle L\rangle}\int dLF(L)\int_0^L dle^{-l/\lambda}\int_l^\infty \frac{1}{\langle l\rangle}\cdot e^{-l'/\langle l\rangle}\,dl'$$

$$= \frac{\lambda^*}{\langle L\rangle}\int (1-e^{-L/\lambda^*})F(L)\,dL = P^*_{esc} \ , \qquad (43)$$

$$P_{tr} = \int dLF(L)e^{-L/\lambda}\int_L^\infty \frac{1}{\langle l\rangle}\,e^{-l'/\langle l\rangle}\,dl'$$

$$= \int dLF(L)e^{-L/\lambda^*} = 1-\frac{\langle L\rangle}{\lambda^*}\,P^*_{esc} \ , \qquad (44)$$

$$P_3 = \int dLF(L)\int_0^L \frac{1}{\langle l\rangle}\cdot e^{-l/\langle l\rangle}e^{-l/\lambda}\,dl$$

$$= C^{s,\infty}\int (1-e^{-L/\lambda^*})F(L)\,dL$$

$$= C^{s,\infty}\frac{\langle L\rangle}{\lambda^*}\,P^*_{esc} \ . \qquad (45)$$

Substitute these results into Eq. (33) to find

$$C^{inter} = \frac{P^*_{esc}\cdot\left(\int H(S)e^{-S/\lambda}\,dS\right)\cdot\left(C^\infty\frac{\langle L\rangle}{\lambda^*}\,P^*_{esc}\right)}{1-\left(\int H(S)e^{-S/\lambda}\,dS\right)\cdot\left(1-\frac{\langle L\rangle}{\lambda^*}\,P^*_{esc}\right)} \ . \qquad (46)$$

Again, it can be seen that the average intervolume Dancoff factor for a fuel kernel is a relatively simple expression involving the first flight escape probability P^*_{esc}, the effective cross section Σ^*, and the chord length PDF $H(S)$ between the finite volumes.

In the following sections, Eq. (46) will be applied to compute the interpebble and intercompact Dancoff factors for pebble bed and prismatic VHTR configurations, respectively.

IV.B.1. Interpebble Dancoff Factors for Pebble Bed Configurations

Equation (22) gives the first flight escape probability P^*_{esc} for a spherical fuel pebble. Substituting this into Eqs. (43), (44), and (45), explicit expressions for P_1, P_{tr}, and P_3 can be found for a fuel kernel in an infinite medium of pebbles:

$$P_1 = \frac{3}{4}\left(\frac{\lambda^*}{R_1}\right)+\frac{3}{4}\left(\frac{\lambda^*}{R_1}\right)^2 e^{-2(R_1/\lambda^*)}$$

$$-\frac{3}{8}\left(\frac{\lambda^*}{R_1}\right)^3\left[1-e^{-2(R_1/\lambda^*)}\right] \ , \qquad (47)$$

$$P_{tr} = -\left(\frac{\lambda^*}{R_1}\right)e^{-2(R_1/\lambda^*)}+\frac{1}{2}\left(\frac{\lambda^*}{R_1}\right)^2\left[1-e^{-2(R_1/\lambda^*)}\right] \ , \qquad (48)$$

and

$$P_3 = C^\infty\left[1+\left(\frac{\lambda^*}{R_1}\right)e^{-2(R_1/\lambda^*)}\right.$$

$$\left.-\frac{1}{2}\left(\frac{\lambda^*}{R_1}\right)^2\left(1-e^{-2(R_1/\lambda^*)}\right)\right] \ . \qquad (49)$$

For P_2, the chord length PDF between two spherical fuel zones of fuel pebbles in an infinite medium is needed. Assuming that the spherical fuel pebbles are randomly distributed in the background moderator, $H(S)$ is expressed as an exponential PDF:

$$H(S) = \frac{1}{\langle S\rangle}\,e^{-S/\langle S\rangle} \ , \quad 0 < S < \infty \ , \qquad (50)$$

where

$$\langle S\rangle = \frac{4R_1(1-FRAC)}{3FRAC}$$

and $FRAC$ is the volume packing fraction of the spherical fuel zone in the whole medium. Substituting Eq. (50) into the expression for P_2, we find

$$P_2 = \int_0^\infty \frac{1}{\langle S\rangle} e^{-S'/\langle S\rangle} e^{-S'/\lambda}\, dS' = \frac{1}{1+\langle S\rangle \dfrac{1}{\lambda}} \quad . \tag{51}$$

It should be noted that the model presented in Eqs. (50) and (51) to calculate P_2 is a crude approximation. To predict accurate interpebble Dancoff factors, a Monte Carlo simulation is needed to obtain P_2. This simulation has been found to be relatively inexpensive to obtain a "1σ" (one relative standard deviation) result accurate to within 10^{-4}. In the results comparison section later in this paper, a comparison will be made between this crude model and the more precise Monte Carlo simulation for calculating P_2.

At this point, expressions for P_1, P_{tr}, P_3, and P_2 have been obtained. Introducing these quantities into Eq. (33), C^{inter} is obtained in terms of basic geometry quantities for fuel pebbles:

$$C^{inter} = C^\infty \frac{\dfrac{3}{4}\dfrac{\lambda^*}{R_1}\left(1+\left(\dfrac{\lambda^*}{R_1}\right)e^{-2(R_1/\lambda^*)} - \dfrac{1}{2}\left(\dfrac{\lambda^*}{R_1}\right)^2[1-e^{-2(R_1/\lambda^*)}]\right)^2 \cdot P_2}{1-P_2\cdot\left(-\left(\dfrac{\lambda^*}{R_1}\right)e^{-2(R_1/\lambda^*)} + \dfrac{1}{2}\left(\dfrac{\lambda^*}{R_1}\right)^2[1-e^{-2(R_1/\lambda^*)}]\right)} \quad , \tag{52}$$

where P_2 is obtained from a simple Monte Carlo calculation.

IV.B.2. Intercompact Dancoff Factors for Prismatic Configurations

To obtain intercompact Dancoff factors for the prismatic VHTR, the fuel compacts will be modeled as both infinite cylinders and finite cylinders.

For both cases, Eqs. (25) and (30) give the first flight escape probabilities respectively. Substituting these formulas into Eqs. (43), (44), and (45), it is straightforward to find P_1, P_{tr}, and P_3 for both infinite and finite cylinders. Note that the average chord length is $\langle L\rangle = 2R_c$ for an infinite cylinder and $\langle L\rangle = 2R_cH/(R_c + H)$ for a finite cylinder.

For P_2, the chord length PDF between two cylindrical fuel compacts is needed. At this point, looking at the definition of P_2 in Eq. (37), it can be seen that this is actually the Dancoff factor definition for finite volumes distributed in a moderator background. So the evaluation of P_2 becomes a classical problem: the calculation of the Dancoff factor for cylindrical fuel rods in a hexagonal lattice, which was the original application studied by Dancoff and Ginsburg[1] in their 1944 report.

In a lattice structure, an accurate expression for the PDF $H(S)$ is difficult to derive analytically, especially for finite cylinders. However, lattice structure, whether square or hexagonal, is a relatively simple geometry for calculating Dancoff factors. Taking advantage of modern-day computing capability, this work can be done in a few seconds using direct Monte Carlo methods, yielding excellent results with a relative standard deviation $<10^{-4}$. Accordingly, Monte Carlo simulation is used to obtain P_2 for infinite and finite cylindrical fuel compacts.

Analytical expressions for P_1, P_{tr}, and P_3 and numerical values for P_2 have now been obtained. Introducing these terms into Eq. (33), either an analytical formula or a numerical estimate for C^{inter} in terms of basic geometry quantities of fuel compacts can be found.

IV.C. Monte Carlo Benchmark Computation

As for Dancoff factors for an infinite and finite medium of fuel kernels, the average Dancoff factors for fuel kernels between fuel pebbles and fuel compacts are calculated by Monte Carlo simulation for comparisons with the results from analytical formulas.

For the interpebble Dancoff factor, the same configuration is set up as for the intrapebble calculation, except that a white boundary condition is set on the outer spherical surface. The packing fraction is fixed at 5.76%, a typical design parameter for VHTR pebble bed reactors, but R_2 is adjusted from 3.0 to 6.0 cm. Neutrons are emitted from the surface of fuel kernels and travel through the fuel zone and moderator shell region. Only those neutrons that reenter the fuel zone at least once and encounter another fuel kernel without colliding with a moderator are tallied. The ratio of this number to the total number of neutrons emitted is the interpebble Dancoff factor.

For the intercompact Dancoff factor, the same hexagonal lattice structure is set up as for the intracompact calculations. For the infinite cylinder, the packing fraction 28.92% is chosen for the intercompact Dancoff factors. The flat-to-flat distance P is adjusted from 2.196 to 4.0 cm. For the finite cylinder, the packing fraction is fixed at 28.9% and H is adjusted from 2 to 100 cm.

IV.D. Numerical Results

In this section, analytical interpebble and intercompact Dancoff factors are compared with Monte Carlo benchmark simulations.

For interpebble Dancoff factors comparisons, the volume packing fraction is fixed at 5.76%. The outer radius

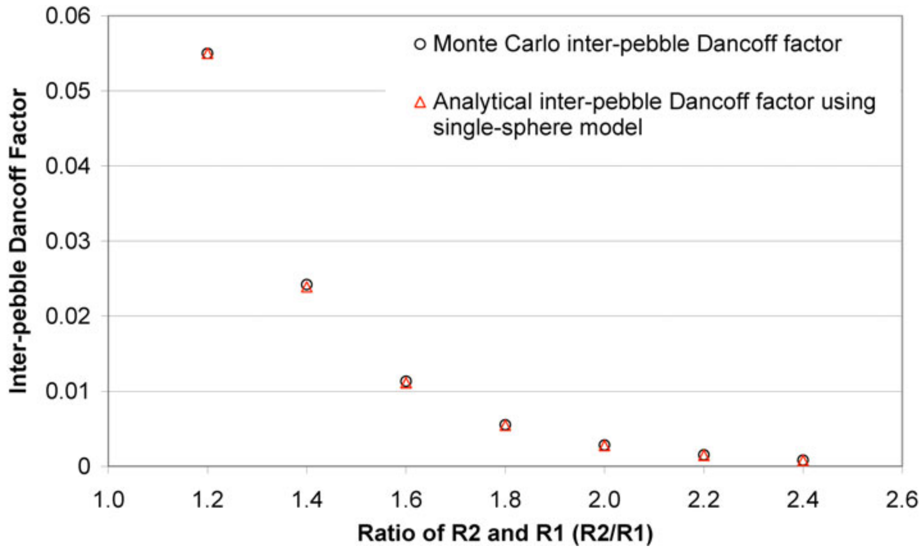

Fig. 10. Comparisons between analytical and Monte Carlo average interpebble Dancoff factors at volume packing fraction 5.76%.

R_2 of the fuel pebble is varied by changing the ratio R_2/R_1 from 1.2 to 2.4. The results are plotted in Fig. 10. It can be seen that the analytical model gives excellent results, with good agreement with Monte Carlo results for different values of R_2/R_1. It should be noted that this agreement is partly due to an accurate calculation of P_2, which is the average probability that a neutron escaping from the fuel zone in a pebble enters the fuel zone in another pebble without collision. A direct Monte Carlo simulation is used to estimate P_2 rather than using the approximate expression given in Eq. (51), which assumes that $H(S)$ is described by an exponential PDF. We have found that Eq. (51) yields poor results for P_2 compared to direct Monte Carlo simulation, as can be seen from Table IV.

Although a white boundary is set on the outer surface of the fuel pebble, which seems expedient for a stochastic distribution of the fuel zones, the assumption of an exponential PDF for $H(S)$ does not give a correct value of P_2 using Eq. (51). When one studies a stochastic distribution of microspheres in a moderator, the assumption of an exponential chord length PDF between microspheres yields excellent results for infinite medium Dancoff factors. However, when one uses the same assumption for fuel zones of pebbles within a full core, inaccurate results are obtained, indicating that the assumption of an exponential chord length PDF between two fuel zones is not appropriate, at least for the problem discussed in this paper.

TABLE IV

Comparison of P_2 from Eq. (51) and Monte Carlo Method

R_2/R_1 (R_1 = 2.5 cm)	P_2 from Eq. (51)	P_2 from Monte Carlo
1.2	0.4990	0.4406
1.4	0.2937	0.1996
1.6	0.1898	0.0945
1.8	0.1305	0.0466
2.0	0.0939	0.0238
2.2	0.0699	0.0125
2.4	0.0535	0.0067

Another analysis can also be given to explain the high accuracy of the single-sphere model results. When $R_2/R_1 = 1.0$, this corresponds to the infinite medium case; i.e., fuel particles are distributed in an infinite background medium. In this case the sum of the intrapebble and interpebble Dancoff factors should equal the infinite medium Dancoff factor, or $(C^{intra} + C^{inter})/C^{\infty} = 1$. The single-sphere model predicts this value exactly when one introduces expressions for C^{intra}, C^{inter}, and C^{∞} into the above relation. Looking at the definitions of C^{intra}/C^{∞}, P_1, P_2, P_3, and P_{tr}, some relations among them should hold. In particular,

$$\frac{C^{intra}}{C^{\infty}} = 1 - P_1 \tag{53}$$

and

$$\frac{P_3}{C^{\infty}} = 1 - P_{tr} \ . \tag{54}$$

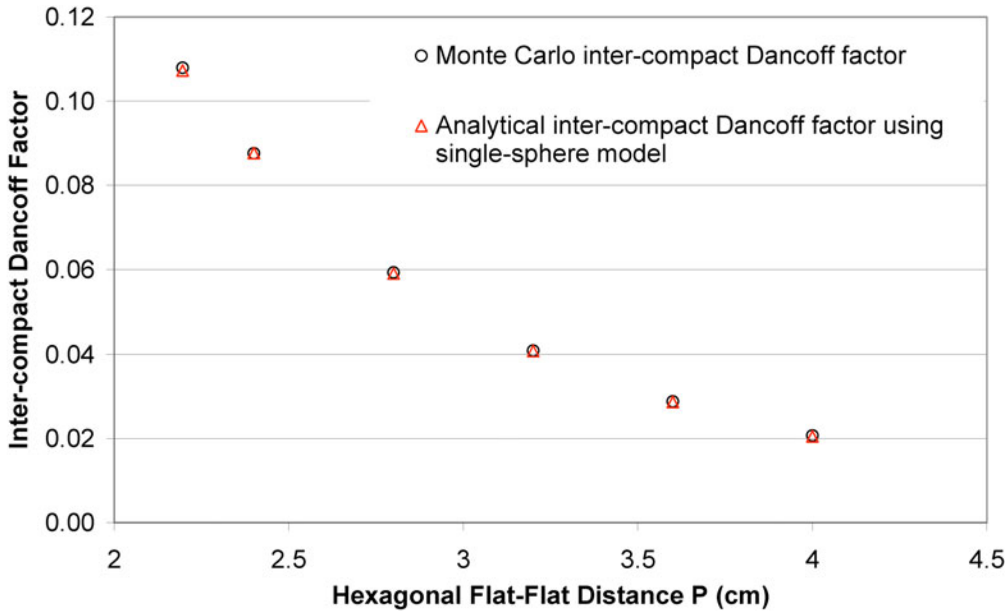

Fig. 11. Comparisons between analytical and Monte Carlo average intercompact Dancoff factors at volume packing fraction 28.92% for infinite height compacts.

The single-sphere model preserves these relations, as shown by inspection of Eqs. (20), (43), (44), and (45). This provides additional assurance for the accuracy of the single-sphere model.

Next, intercompact Dancoff factors are calculated by analytical formulas and compared with the Monte Carlo simulation results. As before, results are given for infinite cylinder and finite cylinder configurations.

IV.D.1. Infinite Cylinder

The volume packing fraction of fuel particles is fixed at 28.92% for the intercompact Dancoff factor computation while the flat-to-flat distance of the hexagonal moderator cell is varied. The results are presented in Fig. 11.

Similar to the interpebble Dancoff factor results, the intercompact Dancoff factors are predicted with excellent accuracy. However, it should be noted that the analytical results calculate P_2 from an inexpensive Monte Carlo simulation.

IV.D.2. Finite Cylinder

For Dancoff factor calculations, the volume packing fraction of fuel particles is fixed at 28.92% within the compact and the height of the hexagonal cell (H) is varied from $2 \times D$ to $100 \times D$, where D is the diameter of the fuel compact.

Results are shown in Fig. 12. It can be seen that when the ratio of the height to the diameter is small, approximately <10, the analytical model gives very poor results, significantly overestimating the intercompact Dancoff factor. This is mainly due to the neutron leakage at the top and bottom surfaces of the fuel compacts. At lower H/D ratios, the leakage is very high, and the neutrons escaping through top and bottom surfaces will not contribute to the intercompact Dancoff factors. However, the analytical model assumes these neutrons can still enter another fuel compact, which overestimates the final results. Once the H/D ratio becomes >20, less leakage occurs, and excellent agreement is obtained with Monte Carlo simulations. In practice, when one analyzes the fuel compact, the ratio is normally about 64 and within the range where the predictions are excellent.

IV.E. Final Remarks

The general methodologies presented in previous Secs. II.A, III.A, and IV.A to calculate Dancoff factors for infinite, finite, and multiple media of fuel lumps have broad application to general reactor physics analysis but are specifically intended for application to the neutronic analysis of TRISO fuel particles in VHTRs. These methodologies can be applied to the analysis of both pebble bed and prismatic VHTR designs because the chord length PDF $F(L)$ is known analytically for spheres[33] and cylinders.[22,34–37] Also, the chord length PDFs $H(S)$ between fuel compacts and between fuel pebbles are well approximated as exponential distributions.[21,39–43] As long as we are able to determine the chord length PDF $f(l)$ between two fuel kernels in an infinite medium, the average Dancoff factor for fuel kernels in either VHTR design is readily obtained by Eqs. (16) and (42).

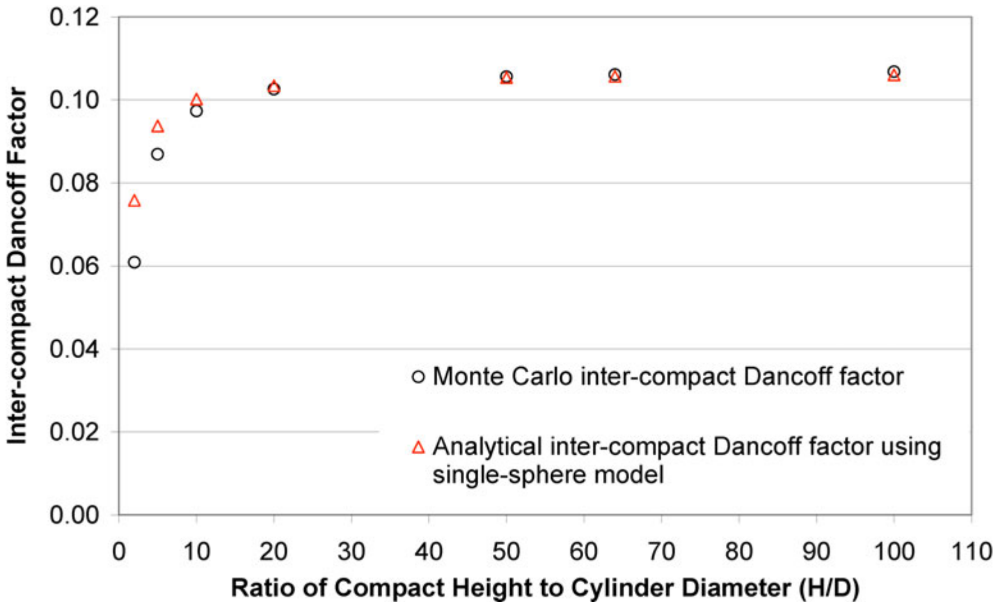

Fig. 12. Comparisons between analytical and Monte Carlo average intercompact Dancoff factors at volume packing fraction 28.92% for finite height compacts.

In Secs. II.D, III.D, and IV.D, numerical results based on the analytical formulas for Dancoff factors are compared with Monte Carlo benchmark results for stochastic configurations generated using the RSA algorithm. Those comparisons were obtained for both pebble bed and prismatic VHTRs.

V. CONCLUSIONS

The chord method was first proposed by Dirac for the calculation of escape probabilities and related quantities more than 60 years ago. Since that time, the chord method has not received much attention outside this specialized area of reactor analysis. The advent of the VHTR as a viable nuclear reactor concept has underscored the importance of accurate and efficient analysis of TRISO fuel configurations characteristic of the prismatic and pebble bed VHTR concepts. The stochastic nature of TRISO fuel coupled with the presence of the double heterogeneity presents substantial computational challenges. In particular, the computation of Dancoff factors for both prismatic and pebble bed reactors is needed for routine neutronic analysis of TRISO fuel configurations. This work demonstrates that the chord method can be used to obtain analytical expressions for Dancoff factors for both prismatic and pebble bed VHTRs.

In this paper, the derivation of analytical expressions for Dancoff factors in stochastic media with double heterogeneities is addressed. General derivations for Dancoff factors in infinite and finite media are obtained by introducing the chord length distribution PDF. The chord length PDF considerably simplifies the mathematical formulas and enables a computationally efficient and accurate approach to calculate Dancoff factors. Theoretical expressions for the chord length PDFs were shown earlier to be accurate and consistent with empirical PDFs obtained with benchmark Monte Carlo simulations. Using these chord length PDFs, the Dancoff factor expressions were applied to infinite and finite VHTR configurations characteristic of both pebble bed and prismatic designs with excellent results, giving results for a range of infinite medium and finite medium Dancoff factors that agreed with benchmark Monte Carlo simulations to well under 1% for pebble bed and prismatic VHTR configurations.

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